Chapter 4

Newtonian Mechanics

4.1. Introduction

The world we live in is a complex place, and we must expect any theory that describes it accurately to share that complexity. But there are three assumptions, satisfied at least approximately in many important physical systems, that together lead to a considerable simplification in the mathematical description of systems for which they are valid.

The first of these assumptions is that the system is "isolated", or "closed", meaning that all forces influencing the behavior of the system are accounted for within the system. The second assumption is that the system is "nonrelativistic", meaning that all velocities are small compared to the speed of light. The third assumption is that the system is "nonquantum", meaning that the basic size parameters of the system are large compared with those of atomic systems (or, more precisely, that the actions involved are large multiples of the fundamental Planck unit of action).

These assumptions put us into the realm of "classical" physics, where dynamical interactions of material bodies are adequately described by the famous three laws of motion of Newton's Principia. Of course, such systems can still exhibit great complexity, and in fact even the famous "three body problem"—to describe completely the motions of three point particles under their mutual gravitational attraction—is still far from "solved". Moreover, at least the latter two of these assumptions are quite sophisticated in nature, and even explaining them carefully requires some doing. Later we shall see that a comparatively unsophisticated fourth assumption—that a system is "close to equilibrium"—cuts through all the complexity and reduces a problem to one that is completely analyzable (using an algorithm called the "method of small vibrations"). This magical assumption, which in effect linearizes the situation, is far from universally valid—magic after all only works on special occasions. But when it does hold, its power is much too valuable to ignore, and we will look at it in some detail at the end of this chapter, after developing the basic theory of Newtonian mechanics and illustrating it with several important examples.

We commence our study of Classical Mechanics with a little history.[1]

4.2. Newton's Laws of Motion

We have already referred several times to "Newton's Laws of Motion". They are a well-recognized milestone in intellectual history and could even be said to mark the beginning of modern physical science, so it is worth looking at them in more detail. They were first published in July of 1686 in a remarkable treatise, usually referred to as Newton's *Principia*,[2] and it is not their mere statement that gives them such importance but rather the manner in which Newton was able to use them in Principia to develop a mathematically rigorous theory of particle dynamics.

Let us look first at Newton's original formulation of his Laws of Motion:

AXIOMATA SIVE LEGES MOTUS

Lex I. Corpus omne perseverare in statuo suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.

[1]We are grateful to Professor Michael Nauenberg of UCSC for his critical reading of this section and for correcting several inaccuracies in these historical remarks.

[2]The full Latin title is "Philosophiae Naturalis Principia Mathematica", or in English, "Mathematical Principles of Natural Philosophy". This first edition is commonly referred to as the 1687 edition, since it was not distributed until a year after it was printed.

Lex II. Mutationem motus proportionalem esse vi motrici impressae, & fieri secundum lineam rectam qua vis illa imprimitur.

Lex III. Actioni contrariam semper & qualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse quales & in partes contrarias dirigi.

Even though we are sure you had no difficulty with the Latin, let's translate that into English:

AXIOMS CONCERNING LAWS OF MOTION

Law 1. Every body remains in a state of rest or of uniform motion in a straight line unless compelled to change that state by forces acting on it.

Law 2. Change of motion is proportional to impressed motive force and is in the same direction as the impressed force.

Law 3. For every action there is an equal and opposite reaction, or, the mutual actions of two bodies on each other are always equal and directed to opposite directions.

The first thing to remark is that, mathematically speaking, there are only **two** independent laws here—the First Law is clearly a special case of the second, obtained by setting the "impressed motive force" to zero.[3]

Another point worth mentioning is that the Second Law does not really say "$F = ma$". Newton was developing the calculus at the same time he was writing the Principia, and no one would have understood his meaning if he had written the Second Law as we do today. In fact, if one reads the Principia, it becomes clear that what Newton intended by the Second Law is something like, "If you strike an object

[3]However, as we shall see later, the First Law does have physical content that is independent of and prior to the Second Law: it asserts the existence of so-called "inertial frames of reference", and it is only in inertial frames that the Second Law is valid. Moreover, the First Law also has great historical and philosophical importance, as we shall explain in more detail at the end of this section.

with a hammer, then the change of its momentum is proportional to the strength with which you hit it and is in the same direction as the hammer moves." That is, Newton is thinking about an instantaneous impulse rather than a force applied continuously over time. So how did Newton deal with a nonimpulsive force that acted over an interval of time, changing continuously as it did so? Essentially he worked out the appropriate differential calculus details each time. That is, he broke the interval into a large number of small subintervals during which the force was essentially constant, applied the Second Law to each subinterval, and then passed to the limit.

The Third Law does **not** say that the force (or "action") that one body exerts on another is directed along the line joining them. However this is how it usually gets used in the Principia, and so it is often considered to be part of the Third Law. We will distinguish between the two versions by referring to them as the weak and strong forms of Newton's Third Law.

It is pretty clear that these Laws of Motion by themselves are insufficient to predict how physical objects will move. What is missing is a specification of what the forces actually are that objects exert upon each other. However, later in the Principia Newton formulated another important law of nature, called The Law of Universal Gravitation. It states that there is an attractive force between any two particles of matter whose magnitude is proportional to the product of their masses and inversely proportional to the square of the distance separating them. One of the most remarkable achievements of the Principia was Newton's derivation of the form of his law of gravitation from the Laws of Motion together with Kepler's laws of planetary motion, and we well give an account of how Newton accomplished this later, after we have developed the necessary machinery.

If one takes Newton's law of gravitation seriously, it would appear that a small movement of a massive object on the Earth would be instantaneously felt as a change in the gravitational force at arbitrarily great distances—say on Jupiter. This "action at a distance" was something that made Newton and many of his contemporaries quite uncomfortable. Today we know that gravitation does **not** work precisely the way that Newton's law suggests. Instead, gravitation is described by a field, and changes in this field propagate with the speed of light. The force on a test particle is not a direct response to the many far-off particles that together generate the field, but rather it is caused by the interaction of the test particle with the gravitational field in its immediate location. To a good approximation, the gravitational field is described by a potential function that gives Newton's law of gravity, but the detailed reality is more complicated, and accounting for small errors observed in certain predictions of Newtonian gravitation requires the more sophisticated theory of Einstein's General Relativity.

Newton's Laws of Motion themselves are now known to be only an approximation. In situations where all the velocities involved are small compared to the speed of light, Newton's Laws of Motion are highly accurate, but at very high velocities one needs Einstein's more refined theory of Special Relativity. Newton's Laws of Motion also break down when dealing with the very small objects of atomic physics. In this realm the more complex Quantum Mechanics is needed to give an accurate description of how particles move and interact.

But even though Newton's Laws of Motion and his Law of Gravitation are not the ultimate description of physical reality, it should not be forgotten that they give an amazingly accurate description of the dynamics of massive objects over a vast range of masses, velocities, and distances. In particular, in the two hundred years following the publication of Principia, the consequences of Newton's Laws of Motion were developed into a mathematical theory of great elegance and power that among other successes made predictions concerning the motions of the planets, moons, comets, and asteroids of our own solar system that were verified with remarkable accuracy. We will cover some of this theory below.

We will end this mainly historical section with an explanation of why the First Law of Motion has such great historical and philosophical importance. We quote Michael Nauenberg (with permission) from part of a private exchange with him on this subject:

> Newton made it clear in the Principia that he credited Galileo with the Second Law. What should be pointed

out is that the great breaktrough in dynamics in Galileo and Newton's time came about with an understanding of the First Law. Before then, it was understood that to initiate motion required an external force, but the idea that motion could be sustained without an external force seems to have escaped attention. Even stones and arrows somehow had to be continuously pushed during their flight by the surrounding air, according to Aristotles and later commentators, until Galileo finally showed that the air only slows them down, and in the absence of air friction, they travelled along a parabolic path. In earlier manuscripts Newton spoke also of "inertial forces". Apparently even he could not free himself completelely from millenia of confusion.

4.3. Newtonian Kinematics

As has become traditional, we will begin our study of the Newtonian worldview with a discussion of the *kinematics* of Newtonian physics, i.e., the mathematical formalism and infrastructure that we will use to describe motion, and only then will we go on to consider the *dynamics*, that is, the nature of the forces that express the real physical content of Newton's theory of motion.

A Newtonian (Dynamical) System (V, F) consists of an orthogonal vector space V, called the *configuration space* of the system, together with a vector field F on V, i.e., a smooth map $F : V \to V$, called the *force law* of the system. (By an orthogonal vector space we just mean a real vector space with a positive definite inner product.) For the time being V will be finite dimensional and its dimension, N, is called the number of degrees of freedom of the system (V, F). Later we will also consider the infinite-dimensional case. If you want to think of V as being $\mathbf{R}^N$ with the usual "dot-product", that is fine, but we will write $\langle u, v \rangle$ to denote the inner product of two elements u and v in V and $\|v\|$ to denote the "length" of a vector v (defined by $\|v\|^2 = \langle v, v \rangle$).

The reason why we call V configuration space is that the points of V are supposed to be in bijective correspondence with all the possible